Anomalous diffusion in a field of randomly distributed scatterers

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We consider the motion of particles which are scattered by randomly distributed obstacles. In between scattering events the particles move uniformly. The governing master equation is obtained by mapping the problem onto a master equation which was previously devised for the description of anomalous diffusion of particles with inertia [R. Friedrich *et al.*, Phys. Rev. Lett. **96**, 230601 (2006)]. We show that for a scale-free distance distribution of scatterers a time-fractional master equation arises. The corresponding diffusion equation which exhibits a power-law diffusion coefficient is solved in d dimensions via the method of subordination.

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I. INTRODUCTION

Many complex systems in nature show significant deviations from the predictions of the standard theory of Brownian diffusion [1–4]. In general the dynamics of such systems cannot be described suitably by the standard theory of Gaussian fluctuations. In many cases the properties of systems exhibiting anomalous diffusion can be related to scale-free probability distributions. For example, the theory of Lévy flights, i.e., random walks with a scale-free distribution of displacement distances, has become a well-established scheme to quantitatively describe superdiffusive dispersion [3]. On the other hand, a common description of subdiffusive processes is in terms of the fractional diffusion equations which can be derived from a continuous time random walk model governed by scale-free waiting time distributions of the random walker [4].

In a recent work we have considered the motion of particles, which are subjected to random scattering events instantaneously changing the particle velocities [5,6]. The occurrences of these impacts are assumed to happen after randomly distributed time intervals. The time in between the impacts is characterized by a waiting time distribution $W(\mathbf{v}|\mathbf{v}';t-t')$, which denotes the probability that two impacts are separated by the interval $t-t'$, changing the velocity from $\mathbf{v}'$ to $\mathbf{v}$. This waiting time distribution can also be regarded as a free flight distribution. Accordingly this process can be considered as a continuous time random walk in velocity space. In [5,6] we have derived the master equation for the probability distribution $f(\mathbf{x},\mathbf{v},t)$ of the particle located at time t at $\mathbf{x}$ moving with constant velocity $\mathbf{v}$. This master equation takes the form

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla_x\right]f(\mathbf{x},\mathbf{v},t) = \int_0^t dt'\int d\mathbf{v}'[Q(\mathbf{v}|\mathbf{v}',t-t')f(\mathbf{x}-\mathbf{v}'(t-t'),\mathbf{v}',t') - Q(\mathbf{v}'|\mathbf{v},t-t')f(\mathbf{x}-\mathbf{v}(t-t'),\mathbf{v},t')]. \tag{1}$$

A derivation of this equation is given in Appendix A. The quantity $Q(\mathbf{v}|\mathbf{v}',t)$ is defined in terms of the waiting time distribution $W(\mathbf{v}|\mathbf{v}',t-t')$ which denotes the joint distribution for the probability that a change of velocity has occurred at time t' where the particle has achieved the velocity $\mathbf{v}'$ and that after the time interval $\tau=t-t'$ the next change happens, where the particle takes on a new velocity $\mathbf{v}$. The kernel $Q(\mathbf{v}|\mathbf{v}',t)$ arising in the master equation is of the same form as the standard kernel of continuous time random walks [7] and can be defined in terms of its Laplace transform

$$Q(\mathbf{v}|\mathbf{v}',\lambda)=\frac{\lambda W(\mathbf{v}|\mathbf{v}',\lambda)}{1-\int d\mathbf{v}W(\mathbf{v}|\mathbf{v}',\lambda)}. \tag{2}$$

We note that the master equation guarantees the conservation of the positiveness of the probability distribution provided the kernel $Q(\mathbf{v},\mathbf{v}',t)$ is related to a waiting time probability distribution $W(\mathbf{v},\mathbf{v}',t)$ obeying the following properties

$$W(\mathbf{v}|\mathbf{v}',\tau)\geq 0,$$

$$\int d\mathbf{v}W(\mathbf{v}|\mathbf{v}',\tau)=1. \tag{3}$$

The choice $Q(\mathbf{v}|\mathbf{v}',t-t')=F(\mathbf{v},\mathbf{v}')\delta(t-t')$ obviously yields a Markovian master equation.

For a scale-free kernel with a power-law dependence on the time interval τ the master equation can be related to the class of fractional master equations which are widely used for the analysis of anomalous diffusion in complex systems [1,2,4]. For the corresponding Fokker-Planck equation, which can be interpreted as a fractional Kramers equation [8], it is possible to derive a description in terms of Langevin equations which can be used to investigate sample trajectories of the process [9]. The subtle differences between the fractional Kramers equation proposed in [5,6] and the one considered in [8] are also discussed in [9].

As we show in the present paper a similar approach can be applied to the problem of a particle freely moving in a field of randomly distributed scatterers. Originally our work has been motivated by numerical simulations of particle motion in magnetohydrodynamic turbulence performed by Homann *et al.* [10]. They found that the particle motion is strongly influenced by the presence of current sheets which are localized structures where an electric current is confined to a surface. Particle trajectories are characterized by long straight flights interrupted by localized events of high acceleration leading to a sudden change of the direction of flight. The occurrence of these events is determined by the spatial distribution of current sheets in magnetohydrodynamic (MHD) turbulence. Assuming the velocity changes to occur almost instantaneously the motion of the particle can therefore be described by a particle flying in a field of scattering current sheets. Within the approximation that the velocity in between two consecutive scattering events remains constant, the statistics of the current sheet distribution can be formulated in terms of waiting time distributions. In turn, the statistical description of the particle's motion can be mapped onto the problem considered in [5,6].

However, our treatment is not limited to a specific problem but is a generic stochastic model for the description of anomalous diffusion of particles in a field of randomly moving scatterers. To state another possible application one can think of a generalized Drude model to describe electrical conduction in dilute gases [11].

In this paper we first formulate the governing master equation by translating the statistics of random distances to random free flight times and thereby mapping the problem to Eq. (1). Then we discuss two important classes of distance distributions of which one leads to the class of fractional equations which is central for the description of anomalous diffusion. Applying a diffusion approximation we state the corresponding generalized diffusion equation and obtain a scaling solution. The necessary background is comprised in two appendixes.

II. MOTION IN A RANDOM FIELD OF SCATTERERS

In this section we shall map the problem of the description of the motion of particles in a field of randomly distributed scatterers onto the master equation (1). We base our treatment on the definition of the probability distribution

$$P(\mathbf{v}|\mathbf{v}';R), \tag{4}$$

which is the spatial analog of the free flight time distribution $W(\mathbf{v}|\mathbf{v}',t-t')$. More specifically, $P(\mathbf{v}|\mathbf{v}',R)dRd\mathbf{v}d\mathbf{v}'$ is the probability that two scatterers are separated by a distance R, whereby the first scatterer changes the particle's velocity to $\mathbf{v}'$ and the second scatterer, after a free flight of the particle, to $\mathbf{v}$, respectively.

In order to apply the master equation (1) which is formulated in terms of free flight times to the description of particle trajectories in a field of randomly distributed scatterers, we have to determine the waiting time distribution $W(\mathbf{v}|\mathbf{v}',\tau)$ from the distance distribution $P(\mathbf{v}|\mathbf{v}',R)$. Introducing the absolute value of the velocity $v=\sqrt{\mathbf{v}^2}$ the particle travels the distance $R=v'(t-t')$ between the two scattering events. The waiting time distribution is therefore obtained according to

$$W(\mathbf{v}|\mathbf{v}';t-t') = \int_0^\infty dRP(\mathbf{v}|\mathbf{v}';R)\delta\left(t-t'-\frac{R}{v'}\right). \tag{5}$$

Using the identity

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \tag{6}$$

of the delta function, where the x_i are the zeros of the function $f(x)$, this yields the explicit relationship

$$W(\mathbf{v}|\mathbf{v}',t-t') = v'P[\mathbf{v}|\mathbf{v}';v'(t-t')]. \tag{7}$$

If we assume the scattering statistics to be statistically independent from the spatial distribution,

$$P(\mathbf{v}|\mathbf{v}',R) = p(\mathbf{v}|\mathbf{v}')g(R) \tag{8}$$

we obtain

$$W(\mathbf{v}|\mathbf{v}',t-t') = v'p(\mathbf{v}|\mathbf{v}')g[v'(t-t')]. \tag{9}$$

Here, $p(\mathbf{v}|\mathbf{v}')d\mathbf{v}d\mathbf{v}'$ is the joint probability relating the velocities $\mathbf{v}'$, $\mathbf{v}$ after the two scattering events, and $g(R)dR$ is the probability finding two scatterers at distance R.

The master equation (1) is formulated in terms of the quantity $Q(\mathbf{v}|\mathbf{v}',t-t')$, which is defined in terms of the Laplace transform by Eq. (2). Applying Eq. (9), one obtains

$$Q(\mathbf{v}|\mathbf{v}',\lambda) = p(\mathbf{v}|\mathbf{v}')\frac{\lambda g\left(\frac{\lambda}{v'}\right)}{1-g\left(\frac{\lambda}{v'}\right)}, \tag{10}$$

where the scaling property of Laplace transforms was used. Inverse Laplace transform finally yields the desired form of the time evolution kernel

$$Q(\mathbf{v}|\mathbf{v}',t-t') = p(\mathbf{v}|\mathbf{v}')v'q[v'(t-t')]. \tag{11}$$

where the Laplace transform of $q(\xi)$ is determined via the Laplace transform of $g(\xi)$

$$q(\lambda) = \frac{\lambda g(\lambda)}{1-g(\lambda)}. \tag{12}$$

Thus we have succeeded in expressing the spatial statistics in terms of waiting time statistics.

Lumping Eq. (11) into Eq. (1) the master equation describing the motion of particles in a field of randomly distributed scatterers in joint position-velocity space reads

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla_x\right]&f(\mathbf{x},\mathbf{v},t)\\ &= \int_0^t dt'\int d\mathbf{v}'\{p(\mathbf{v}|\mathbf{v}')v'q[v'(t-t')]\\ &\quad - \delta(\mathbf{v}-\mathbf{v}')vq[v(t-t')]\}f[\mathbf{x}-\mathbf{v}'(t-t'),\mathbf{v}',t'].\end{aligned} \tag{13}$$

This master equation has a straightforward interpretation. The first summand in the bracket on the right-hand side indicates that the probability to be at time t at position $\mathbf{x}$ with velocity $\mathbf{v}$ is increased by particles starting at $\mathbf{x}'=\mathbf{x}-\mathbf{v}'(t-t')$ with velocity $\mathbf{v}'$ and performing after a free flight of duration $t-t'$ at $\mathbf{x}$ a transition to the velocity $\mathbf{v}$. On the other hand the second term in the bracket evidences that the probability is decreased by particles which perform at $\mathbf{x}$ after a free flight time of duration $t-t'$ a transition away from $\mathbf{v}$ to some other velocity.

In the following we shall consider various forms of the quantities $p(\mathbf{v}|\mathbf{v}')$ and waiting time distributions defining the kernel $q(\xi)$. Thereby we shall focus on the corresponding probability distribution $h(\mathbf{v},t)$ for the velocity of the particle

$$h(\mathbf{v},t)=\int d\mathbf{x}f(\mathbf{x},\mathbf{v},t). \tag{14}$$

Integrating Eq. (52) with respect to the particle coordinate $\mathbf{x}$ we obtain a generalized master equation in velocity space

$$\frac{\partial}{\partial t}h(\mathbf{v},t)=\int_0^t dt'\int d\mathbf{v}'\{p(\mathbf{v}|\mathbf{v}')v'q[v'(t-t')] - \delta(\mathbf{v}-\mathbf{v}')vq[v(t-t')]\}h(\mathbf{v}',t'). \tag{15}$$

We can consider the Laplace transform of this equation. It reads

$$\lambda h(\mathbf{v},\lambda)=h(\mathbf{v},0)+\int d\mathbf{v}'\left[p(\mathbf{v}|\mathbf{v}')q\left(\frac{\lambda}{v'}\right) - \delta(\mathbf{v}-\mathbf{v}')q\left(\frac{\lambda}{v'}\right)\right]h(\mathbf{v}',\lambda). \tag{16}$$

In the following we consider two generic forms of waiting time distributions, namely exponential and power law distributed waiting times. Eq. (16) will be especially convenient for the later.

III. WAITING TIME DISTRIBUTIONS AND STATISTICS OF SCATTERERS

Let us now consider several examples for the spatial statistics of the distance R between the scatterers. In the following d denotes the dimension of space. Our first example is the exponential probability distribution

$$g(R)=\gamma R^{d-1}e^{-\gamma R}. \tag{17}$$

To calculate the evolution kernel q we have to switch to Laplace space. We obtain

$$g(\lambda)=\frac{\gamma\Gamma(d)}{(\lambda+\gamma)^d}, \tag{18}$$

where Γ denotes the well-known Gamma function. The function $q(\lambda)$ then takes the form

$$q(\lambda)=\frac{\gamma\lambda\Gamma(d)}{(\lambda+\gamma)^d-\gamma\Gamma(d)}. \tag{19}$$

The inverse Laplace transform of Eq. (19) in arbitrary dimensions cannot be stated in a closed form. Let us therefore consider two special cases. In one dimension the inverse Laplace transform of Eq. (19) reads

$$q(\xi)=\gamma\delta(\xi) \tag{20}$$

leading to a Markovian master equation for the velocity

$$\frac{\partial}{\partial t}h(v,t)=\gamma\int dv'[p(v|v')-\delta(v-v')]h(v',t'). \tag{21}$$

In two dimensions the kernel takes the form

$$q(\xi)=\gamma e^{-\gamma\xi}(\cosh[\sqrt{\gamma}\xi]-\sqrt{\gamma}\sinh[\sqrt{\gamma}\xi]). \tag{22}$$

Insertion of this expression into the master equation (15) yields the desired equation in two dimensions. This equation simplifies significantly if we consider the case $\gamma=1$. Then Eq. (22) reads

$$q(\xi)=e^{-2\xi} \tag{23}$$

and we obtain the master equation

$$\frac{\partial}{\partial t}h(\mathbf{v},t)=\int_0^t dt'\int d\mathbf{v}'[p(\mathbf{v}|\mathbf{v}')v'e^{-2v'(t-t')} - \delta(\mathbf{v}-\mathbf{v}')ve^{-2v(t-t')}]h(\mathbf{v}',t'). \tag{24}$$

In more than two dimensions even the special case of $\gamma=1$ leads to rather lengthy expressions.

As a second example we want to consider the important class of scale-free probability distributions $g(R)$ which play a prominent role in the theory of anomalous diffusion [4]. These are characterized by an algebraic decay in the limit $R\to\infty$. In d dimensions they take the generic form

$$g(R)\approx R^{d-1}\frac{1}{R^{\gamma}}. \tag{25}$$

In order to lack a typical length scale, the first moment of these distributions has to diverge

$$\langle R\rangle=\int_0^{\infty}dRg(R)R\to\infty. \tag{26}$$

This means that the integral diverges for large values of R

$$\int^R dR'R'^{d-1}\frac{1}{R'^{\gamma}}R'=\int^R dR'\frac{1}{R'^{\gamma-d}}. \tag{27}$$

Introducing the dimension dependent exponent $\alpha=\gamma-d$, we observe that the distribution is scale free for $0<\alpha\leq 1$.

Note that the distributions under consideration belong to the class of spherically symmetric d-dimensional Lévy stable probability distributions, denoted $p_{\alpha}^{d}(R)$, which display a large R asymptotic behavior of the form $p_{\alpha}^{d}(R)\approx R^{-(\alpha+d)}=R^{-\gamma}$ which are scale free for $0<\alpha\leq 1$ (see, e.g., [12]).

To establish the kernel $vq[v(t-t')]$ for this case we have to consider the Laplace transform again. According to the Tauberian theorems the large R behavior is determined by the behavior of the Laplace transform $g(\lambda)$ of the probability distribution $g(R)$, i.e., its characteristic function, at small values of λ [13]. For the considered distributions with algebraic decay we can expand the characteristic function for small λ according to

$$g(\lambda) \approx 1 - R_D^{\alpha}\lambda^{\alpha} + \cdots . \tag{28}$$

In order that the distribution is scale free, we have to demand that $g(\lambda)$ is not differentiable at $\lambda=0$, which indicates that the first moment diverges. We remind the reader that

$$\langle R\rangle = -\left[\frac{\partial}{\partial\lambda}g(\lambda)\right]_{\lambda=0}. \tag{29}$$

This restricts the value of α again to the interval $0<\alpha<1$. Next we have evaluate the quantity

$$q(\lambda) = \frac{\lambda g(\lambda)}{1-g(\lambda)} = \lambda\frac{1-(R_D\lambda)^{\alpha}}{(R_D\lambda)^{\alpha}} \approx R_D^{-\alpha}\lambda^{1-\alpha}. \tag{30}$$

Inverse Laplace transform yields an approximation for the kernel $vq[v(t-t')]$ for large values of $v(t-t')$. As we will now indicate this leads to the introduction of fractional derivatives.

We consider the convolution

$$\int_0^t \frac{dt'}{(t-t')^{1-\alpha}}G(t') \tag{31}$$

whose Laplace transform is simply

$$\Gamma(\alpha)\lambda^{-\alpha}G(\lambda), \tag{32}$$

where $\Gamma(\alpha)$ denotes the Gamma function and the relation holds for $\alpha>0$. If we now consider the product

$$\lambda^{1-\alpha}G(\lambda) \tag{33}$$

in Laplace space, we can represent this in real space as the following integral

$$\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{dt'}{(t-t')^{1-\alpha}}G(t') = D_t^{1-\alpha}G(t), \tag{34}$$

which defines the Riemann-Liouville fractional derivative. For the discussion of the fractional derivative we refer the reader to, e.g., [14].

In fact, we are interested in the expression

$$q\left(\frac{\lambda}{v}\right)G(\lambda) = v^{\alpha-1}\lambda^{1-\alpha}G(\lambda), \tag{35}$$

which in real space yields

$$v^{\alpha-1}D_t^{1-\alpha}G(t). \tag{36}$$

After insertion of the kernel (35) into Eq. (16) we can now formulate a fractional master equation for a scale-free distribution of distances between the scatterers

$$\frac{\partial}{\partial t}h(\mathbf{v},t) = D_t^{1-\alpha}\int d\mathbf{v}'[p(\mathbf{v}|\mathbf{v}')v'^{\alpha-1} - \delta(\mathbf{v}-\mathbf{v}')v^{\alpha-1}]h(\mathbf{v}',t). \tag{37}$$

It is quite interesting to notice how our approach connects Lévy distributed scattering distances with a time-fractional master equation for the velocity distribution.

IV. FRACTIONAL EQUATIONS AND VELOCITY DISTRIBUTION

In this section we shall construct the solution for the fractional master equation (37). For convenience let us first define the operator $L(\mathbf{v},\mathbf{v}')$

$$L(\mathbf{v},\mathbf{v}') = [p(\mathbf{v}|\mathbf{v}')v'^{\alpha-1} - \delta(\mathbf{v}-\mathbf{v}')v^{\alpha-1}] \tag{38}$$

since we can thereby cast the master equation into the form

$$\frac{\partial}{\partial t}h(\mathbf{v},t) = D_t^{1-\alpha}\int d\mathbf{v}'L(\mathbf{v},\mathbf{v})h(\mathbf{v}',t). \tag{39}$$

A well-known method to solve fractional master equations is by applying an integral transform which maps the solution of the ordinary equation to the corresponding solution of the fractional equation [15]

$$f(\mathbf{v},t) = \int_0^{\infty} ds p(s,t) f_0(\mathbf{v},s), \tag{40}$$

where $f_0(\mathbf{v},s)$ is the solution of the ordinary equation

$$\frac{\partial}{\partial s}f_0(\mathbf{v},s) = \int d\mathbf{v}'L(\mathbf{v},\mathbf{v}')f_0(\mathbf{v}',s). \tag{41}$$

The quantity $p(s,t)$ has the meaning of a probability density. As we show in Appendix B, this distribution is characterized by

$$\frac{\partial}{\partial t}p(s,t) = -\int_0^t dt' q_{1-\alpha}(t-t')\frac{\partial}{\partial s}p(s,t), \tag{42}$$

where $q_{1-\alpha}(t-t')$ is specific time kernel. For the case of fractional equations this equation can be solved analytically and yields

$$p(s,t) = \frac{1}{\alpha}\frac{t}{s^{1+1/\alpha}}L_{\alpha}\left(\frac{t}{s^{1/\alpha}}\right). \tag{43}$$

Here, $L_{\alpha}(x)$ denotes the one-sided Lévy stable distribution of order α [15]. The distribution (43) plays a central role in the theory of fractional equations and is often referred to as *inverse Lévy distribution*.

Interestingly the solution of fractional equations via integral transforms is closely related to the mathematical concept of subordination [16–18]. One can think of the variable s to denote a form of *internal time*, which is randomly mapped to the *physical time* t. This map is specified by a stochastic process

$$s = S(t). \tag{44}$$

In this context $p(s,t)$ denotes the probability distribution to find the internal time s at physical time t. The master equation (41) specifies a Markovian stochastic process in internal time $\mathbf{v}(s)$ and the process under consideration is constructed by subordination, i.e.,

$$\mathbf{v}(t) = \mathbf{v}[S(t)]. \tag{45}$$

Let us summarize. For the case of a self-similar distribution $g(R)$, the determination of the probability distribution of the scattered particle can be reduced to the determination of the probability distribution $f_0(\mathbf{v},s)$ of a Markovian process and a subsequent evaluation of the integral transform (40)

$$f(\mathbf{v},t) = \int_0^\infty ds \frac{1}{\alpha} \frac{t}{s^{1+1/\alpha}} L_\alpha\left(\frac{t}{s^{1/\alpha}}\right) f_0(\mathbf{v},s). \tag{46}$$

The presented method is especially useful if the solution of the ordinary master equation has a scaling form with scaling exponent η, i.e.,

$$f_0(\mathbf{v},s) = \frac{1}{s^{d\eta}} \tilde{F}\left(\frac{\mathbf{v}}{s^\eta}\right) \tag{47}$$

A straightforward calculation then shows that the probability distribution $f(\mathbf{v},t)$ exhibits a scaling behavior with scaling exponent $\zeta = \alpha\eta$,

$$\begin{aligned} f(\mathbf{v},t) &= \int_0^\infty ds \frac{1}{\alpha} \frac{t}{s^{1+1/\alpha}} L_\alpha\left(\frac{t}{s^{1/\alpha}}\right) \frac{1}{s^{d\eta}} \tilde{F}\left(\frac{\mathbf{v}}{s^\eta}\right) \\ &= \int_0^\infty \frac{d\tilde{s}}{\tilde{s}^{1+1/\alpha}} \frac{1}{\tilde{s}^{d\eta}} \frac{1}{t^{d\alpha\eta}} \tilde{F}\left(\frac{\mathbf{v}}{\tilde{s}^\eta t^{\alpha\eta}}\right) = \frac{1}{t^{d\zeta}} F\left(\frac{\mathbf{v}}{\tilde{s}^\eta t^\zeta}\right). \end{aligned} \tag{48}$$

In the following we shall discuss the determination of the probability distribution $f_0(\mathbf{v},s)$ for the diffusion approximation of the transition probability $p(\mathbf{v},\mathbf{v}')$.

V. DIFFUSION APPROXIMATION

Since master equations are hard to tackle analytically for general transition amplitudes, we shall consider now the diffusion approximation of the velocity jump process under consideration. This approximation holds for specific case where the scatterers change the velocity only slightly corresponding to almost elastic small angle scattering. Employing the appropriate diffusion approximation for the velocity transition amplitude we can write [19,20]

$$p(\mathbf{v}|\mathbf{v}')_\beta = \delta(\mathbf{v}-\mathbf{v}')[1 + \beta\Delta_{v'} + \mathcal{O}(\beta^2)], \tag{49}$$

where Δ_v denotes the Laplace operator in velocity space. This formal representation is obtained in case that a Gaussian transition amplitude

$$p(\mathbf{v}|\mathbf{v}') = \frac{1}{\sqrt{4\pi\beta^d}} e^{-(\mathbf{v}-\mathbf{v}')^2/4\beta} \tag{50}$$

in the limit $\beta \to 0$ is considered [19]. Inserting the transition amplitude (50) into the master equation (13) and demanding simultaneously the limit

$$q(R) = \frac{1}{\beta}\tilde{q}(R) \tag{51}$$

to be well defined we obtain

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{v},t) \\ &= \int_0^t dt' \Delta_v v \tilde{q}(v(t-t')) f(\mathbf{x}-\mathbf{v}(t-t'),\mathbf{v},t'). \end{aligned} \tag{52}$$

Equation (52) is a generalized Fokker-Planck equation determining the joint position-velocity distribution of diffusing particles in a field of randomly moving scatterers that change the velocity of the particle only slightly.

It is now interesting to consider the probability distribution of the velocity only. Integrating Eq. (52) with respect to the particle coordinate $\mathbf{x}$ we obtain a generalized diffusion equation for the velocity

$$\frac{\partial}{\partial t} h(\mathbf{v},t) = \int_0^t dt' \Delta_v v \tilde{q}[v(t-t')] h(\mathbf{v},t'). \tag{53}$$

Note that the nonuniformly distributed time intervals between the scattering events lead to a time-dependent diffusion coefficient which is furthermore nonlocal in time.

Let us now consider the case, where the distances between two consecutive scattering events are governed by a spherically symmetric Lévy distribution of order α. According to the last chapter we have to consider a scale-free kernel for which we obtain the fractional equation

$$\frac{\partial}{\partial t} h(\mathbf{v},t) = D_t^{1-\alpha} \Delta_v v^{\alpha-1} h(\mathbf{v},t). \tag{54}$$

As outlined in the previous section this equation can be solved by an integral transform related to the method of subordination. According to Eqs. (40) and (41), we have to solve the ordinary equation for the probability distribution $h_0(\mathbf{v},s)$ determined by the Fokker-Planck equation with respect to the internal time s,

$$\frac{\partial}{\partial s} h_0(\mathbf{v},s) = \Delta_v v^{\alpha-1} h_0(\mathbf{v},s). \tag{55}$$

A solution of this equation in one dimension has been stated in [21]. However, this equation can also be solved in d dimensions for isotropic initial conditions. Transforming to spherical coordinates the determining equation for h_0 then reads

$$\frac{\partial}{\partial s} h_0(v,s) = \frac{1}{v^{d-1}} \frac{\partial}{\partial v} v^{d-1} \frac{\partial}{\partial v} v^{\alpha-1} h_0(v,s). \tag{56}$$

To find a solution of Eq. (56) we perform the scaling ansatz

$$h_0 = \frac{1}{\psi^d(s)} H(\xi), \tag{57}$$

where the transformation $\xi = \frac{v}{\psi(s)}$ has been introduced. Inserting this scaling ansatz, Eq. (56) is transformed to

$$\frac{\dot{\psi}(s)}{\psi^{\alpha-2}(s)} \frac{1}{\xi^{d-1}} \frac{d}{d\xi} \xi^d H(\xi) = \frac{1}{\xi^{d-1}} \frac{d}{d\xi} \xi^{d-1} \frac{d}{d\xi} \xi^{\alpha-1} H(\xi). \tag{58}$$

Separation of variables then yields the determining equation for $\psi(s)$

$$\frac{\dot{\psi}(s)}{\psi^{\alpha-2}(s)} = C, \tag{59}$$

where C is some constant factor. This equation can be solved

$$\psi(s) = \left[(\alpha - 3)\left(\frac{\psi_0^{3-\alpha}}{\alpha - 3} - Cs \right) \right]^{1/3-\alpha}, \tag{60}$$

where $\psi_0 = \psi(0)$ is the initial condition. Without loss of generality we set $\psi(0)=0$ and obtain

$$\psi(s) = [C(3-\alpha)s]^{1/3-\alpha}. \tag{61}$$

To solve the equation for $H(\xi)$ we observe from Eq. (58) that the determining equation reads

$$-CH(\xi) = \frac{1}{\xi}\frac{d}{d\xi}\xi^{\alpha-1}H(\xi), \tag{62}$$

where the remaining integration constant has been set to zero. We then obtain

$$H(\xi) = \frac{Q}{\xi^{\alpha-1}}\exp\left[-C\frac{\xi^{3-\alpha}}{3-\alpha} \right], \tag{63}$$

where Q is some integration constant. Finally, we employ the scaling ansatz (57) to obtain

$$h_0(v,s) = N\frac{v^{1-\alpha}}{[(3-\alpha)s]^{(d+1-\alpha)/(3-\alpha)}}\exp\left[-\frac{v^{3-\alpha}}{(3-\alpha)^2 s} \right], \tag{64}$$

where N is a suitable normalization constant.

Observe that for $\alpha=1$ Eq. (64) reduces to the solution of an isotropic d-dimensional diffusion process as it should. However, even the ordinary diffusion equation without subordination already shows for $0<\alpha<1$ anomalous behavior. Calculating the behavior of the second order moment

$$\langle v(s)^2 \rangle \sim s^{2/3-\alpha}, \tag{65}$$

we observe a subdiffusive behavior for the α-range of interest.

As we have discussed above, the scaling behavior of the probability distributions $p(s,t)$ as well as scaling behavior of the function $h_0(v,s)$ infers the scaling behavior of the probability distribution of $h(v,t)$. Applying the relation (48) we obtain for the present case

$$h(v,t) = \frac{1}{t^{d\alpha/(3-\alpha)}}F\left(\frac{v}{t^{\alpha/(3-\alpha)}} \right). \tag{66}$$

This implies that velocity moments scale like

$$\langle v(t)^n \rangle \sim t^{n\alpha/(3-\alpha)}. \tag{67}$$

It is tempting to consider our diffusion approximation as a Itô interpretation of an underlying stochastic process, while the Hänggi-Klimontovich (postpoint) interpretation would lead to an fractional diffusion equation of the form [22]

$$\frac{\partial}{\partial t}h(\mathbf{v},t) = D_t^{1-\alpha}\nabla_v v^{\alpha-1}\nabla_v h(\mathbf{v},t). \tag{68}$$

The underlying ordinary diffusion equation is for this case just the equation describing diffusion on fractal structures to which the solution is well known [23]. Interestingly, a fractional equation of the form (68) arises in the context of Langrangian fluid turbulence as well [24].

VI. CONCLUSIONS

We have discussed a generic stochastic model describing anomalous diffusion of particles in a field of randomly moving scatterers. The governing master equation of this model has been derived by mapping the problem onto the description of anomalous diffusion of inertial particles. Two natural forms of distributions for the distance traveled between two consecutive scattering events have been considered. For a scale-free distance distribution a time-fractional master equation could be established. This is particularly interesting since this equation connects Lévy distributed distances, commonly described by space-fractional equations, with the class of time-fractional equations. We have outlined how this equation can in principal be solved by an integral transformation and the related method of subordination. For the diffusion approximation of the master equation, we have obtained the corresponding fractional Fokker-Planck equation. We have established a d-dimensional fractional diffusion equation with a power-law diffusion coefficient for the velocity. For the corresponding ordinary (nonfractional) equation a scaling solution has been derived. By subordination the scaling form of the fractional equation has then been devised.

This work has been motivated by numerical investigations of MHD turbulence which show that tracer particles are strongly influenced by localized structures leading to a sudden change of the direction of flight. Due to the general character of the model, one can also think other applications. If for example an external field is incorporated, it could be possible to construct a generalized Drude model for electrical conductance in dilute disordered media. Concluding we would like to point out that the solution of the d-dimensional fractional diffusion equation with a power-law diffusion coefficient is of interest itself (see [21] and references therein).

APPENDIX A: DERIVATION OF THE MASTER EQUATION

In the following we shall derive the master equation. To this end we introduce the probability distribution $\eta(\mathbf{x},\mathbf{v},t)$. This distribution is related to the probability having arrived at time t at the infinitesimal volume element $d\mathbf{x}$ close to $\mathbf{x}$ achieving the new velocity $\mathbf{v}$ from the interval $d\mathbf{v}$

$$\eta(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}. \tag{A1}$$

This probability can be related to the quantity

$$W(\mathbf{v}|\mathbf{v}',t-t')dt'd\mathbf{v}' \tag{A2}$$

which gives the probability that the particle has achieved the velocity $\mathbf{v}'$ from the interval $d\mathbf{v}'$ at time t and has changed this velocity at time t to the velocity $\mathbf{v}'$. It is related to the waiting time distribution

$$W_T(t-t') = \int d\mathbf{v}W(\mathbf{v}|\mathbf{v}',t-t'). \tag{A3}$$

The introduction of this distribution immediately allows us to establish the relationship

$$\eta(\mathbf{x},\mathbf{v},t) = \int_0^t dt' \int d\mathbf{x}' \int d\mathbf{v}' W(\mathbf{v}|\mathbf{v}',t-t') \times \delta[\mathbf{x}-\mathbf{x}'-\mathbf{v}'(t-t')]\eta(\mathbf{x}',\mathbf{v}',t') + f(\mathbf{x},\mathbf{v},0)\delta(t), \tag{A4}$$

where the last term accounts for the initial condition. It describes the change of the probability $\eta(\mathbf{x},\mathbf{v},t)$ between two consecutive changes of the velocity at times t' and t. At time t' the particle has achieved the velocity $\mathbf{v}'$ and has changed this velocity at time t to $\mathbf{v}$. We have taken into account that the particle has traveled during the waiting time $t-t'$ from $\mathbf{x}'$ to

$$\mathbf{x} = \mathbf{x}' + \mathbf{v}'(t-t'). \tag{A5}$$

We are interested in the probability distribution $f(\mathbf{x},\mathbf{v},t)$. This distribution is obtained from $\eta(\mathbf{x}',\mathbf{v},t')$ according to

$$f(\mathbf{x},\mathbf{v},t) = \int d\mathbf{x}' \int_0^t dt' w(t-t') \times \delta[\mathbf{x}-\mathbf{x}'-\mathbf{v}(t-t')]\eta(\mathbf{x}',\mathbf{v},t'). \tag{A6}$$

Thereby, we have introduced the probability

$$w(t) = 1 - \int_0^t dt' \int d\mathbf{v} W(\mathbf{v},\mathbf{v}',t-t') = 1 - \int_0^t dt' W_T(t-t') \tag{A7}$$

that no transition of the velocity of the particle is observed in the interval $t-t'$.

In the following we shall combine both equations, Eq. (A4) and Eq. (A6) in order to obtain the desired master equation for $f(\mathbf{x},\mathbf{v},t)$.

From Eq. (A6) we obtain by taking the time derivative

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{v},t) = \eta(\mathbf{x},\mathbf{v},t) + \int_0^t dt' \int d\mathbf{x}' \frac{d}{dt} w(t-t') \times \delta[\mathbf{x}-\mathbf{x}'-\mathbf{v}(t-t')]\eta(\mathbf{x}',\mathbf{v},t'). \tag{A8}$$

Due to the relationship

$$\frac{d}{dt} w(t-t') = -\int d\mathbf{v} W(\mathbf{v}|\mathbf{v}',t-t') \tag{A9}$$

we arrive with Eq. (A4) at

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{v},t) = \int_0^t dt' \int d\mathbf{v}' W(\mathbf{v}|,\mathbf{v}',t-t') e^{-\mathbf{v}'\cdot\nabla_x(t-t')}\eta(\mathbf{x}',\mathbf{v}',t') - \int_0^t dt' \int d\mathbf{v}' W(\mathbf{v}'|,\mathbf{v},t-t') e^{-\mathbf{v}\cdot\nabla_x(t-t')}\eta(\mathbf{x}',\mathbf{v},t') + f(\mathbf{x},\mathbf{v},0), \tag{A10}$$

where we have introduced the free streaming operator

$$\int d\mathbf{x}' \delta(\mathbf{x}-\mathbf{x}'-\mathbf{v}(t-t'))f(\mathbf{x}',t') = e^{-\mathbf{v}\cdot\nabla_x(t-t')}f(\mathbf{x},t'). \tag{A11}$$

A closed master equation is obtained by eliminating $\eta(\mathbf{x},\mathbf{v},t)$. This can be achieved by considering Eq. (A4) in the form

$$f(\mathbf{x},\mathbf{v},t) = \int_0^t dt' w(t-t') e^{-\mathbf{v}\cdot\nabla_x(t-t')}\eta(\mathbf{x},\mathbf{v},t'). \tag{A12}$$

This equation can be solved introducing the Laplace transforms

$$f(\mathbf{x},\mathbf{v},\lambda) = \int_0^\infty dt e^{-\lambda t} f(\mathbf{x},\mathbf{v},t),$$

$$\eta(\mathbf{x},\mathbf{v},\lambda) = \int_0^\infty dt e^{-\lambda t} \eta(\mathbf{x},\mathbf{v},t),$$

$$w(\lambda) = \int_0^\infty dt e^{-\lambda t} w(t). \tag{A13}$$

This yields the operator equation

$$f(\mathbf{x},\mathbf{v},\lambda) = w(\lambda + \mathbf{v}\cdot\nabla_x), \eta(\mathbf{x},\mathbf{v},\lambda) \tag{A14}$$

which can be inverted

$$\eta(\mathbf{x},\mathbf{v},\lambda) = [w(\lambda + \mathbf{v}\cdot\nabla_x)]^{-1} . f(\mathbf{x},\mathbf{v},\lambda). \tag{A15}$$

This can be expressed as a convolution in real space,

$$\eta(\mathbf{x},\mathbf{v},t) = \int_0^t dt' w^{-1}(t-t') e^{-\mathbf{v}\cdot\nabla_x(t-t')} f(\mathbf{x},\mathbf{v},t'). \tag{A16}$$

The Laplace transform of $w^{-1}(t)$ is simply

$$\frac{1}{w(\lambda)} = \frac{\lambda}{1 - \int d\mathbf{v}' W(\mathbf{v}|\mathbf{v}',\lambda)}. \tag{A17}$$

Inserting this relation into Eq. (A10) we obtain a closed evolution equation for the probability distribution $f(\mathbf{x},\mathbf{v},t)$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{v},t) = \int_0^t dt' \int d\mathbf{x}' \int d\mathbf{v}' W(\mathbf{v}|,\mathbf{v}',t-t') \times \int_0^{t'} dt'' w^{-1}(t'-t'') e^{-\mathbf{v}'\cdot\nabla_x(t-t'')} f(\mathbf{x},\mathbf{v}',t'') - \int_0^t dt' \int d\mathbf{x}' \int d\mathbf{v}' W(\mathbf{v}'|,\mathbf{v},t-t') \times \int_0^{t'} dt'' w^{-1}(t'-t'') e^{-\mathbf{v}\cdot\nabla_x(t-t'')} f(\mathbf{x},\mathbf{v},t''). \tag{A18}$$

Now, we can introduce the quantity

$$Q(\mathbf{v}|\mathbf{v}',t)=\int_0^t dt' W(\mathbf{v}|\mathbf{v}',t-t')w^{-1}(t'). \quad \text{(A19)}$$

This convolution can be evaluated in Laplace space, which yields

$$Q(\mathbf{v}|\mathbf{v}',\lambda)=W(\mathbf{v}|\mathbf{v}',\lambda)[w(\lambda)]^{-1}=\frac{\lambda W(\mathbf{v}|\mathbf{v}',\lambda)}{1-\int d\mathbf{v}' W(\mathbf{v}'|\mathbf{v},\lambda)}. \quad \text{(A20)}$$

This leads us to the final version of the master equation

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla_x\right]f(\mathbf{x},\mathbf{v},t) = \int_0^t dt'\int d\mathbf{x}'\int d\mathbf{v}' Q(\mathbf{v}|,\mathbf{v}',t-t') \times\delta[\mathbf{x}-\mathbf{x}'-\mathbf{v}'(t-t')]f(\mathbf{x}',\mathbf{v}',t') - \int_0^t dt'\int d\mathbf{x}'\int d\mathbf{v}' Q(\mathbf{v}'|,\mathbf{v},t-t') \times\delta[\mathbf{x}-\mathbf{x}'-\mathbf{v}(t-t')]f(\mathbf{x}',\mathbf{v},t'). \quad \text{(A21)}$$

APPENDIX B: SUBORDINATION

We consider the solution of the generalized Fokker-Planck equation

$$\frac{\partial}{\partial t}h(v,t)=\int_0^t dt' q(t-t')Lh(v,t'), \quad \text{(B1)}$$

where L is a Fokker-Planck operator. We shall construct this solution in terms of the integral transform

$$h(v,t)=\int_0^\infty ds p(s,t)h_0(v,s), \quad \text{(B2)}$$

where we assume

$$\frac{\partial}{\partial s}h_0(v,s)=Lh_0(v,s). \quad \text{(B3)}$$

Inserting Eq. (B2) into Eq. (B1) we obtain with Eq. (B3)

$$\frac{\partial}{\partial t}\int dsp(s,t)h_0(v,s)=\int_0^t dt'\int dsq(t-t')p(s,t')Lh_0(v,s) = \int ds\int_0^t dt' q(t-t')p(s,t')\frac{\partial}{\partial s}h_0(v,s). \quad \text{(B4)}$$

Partial integration yields

$$\frac{\partial}{\partial t}\int dsp(s,t)h_0(v,s) = -\int ds\int_0^t dt' q(t-t')\frac{\partial}{\partial s}p(s,t')h_0(v,s) - \int_0^t dt' q(t-t')p(0,t')h_0(v,0). \quad \text{(B5)}$$

The Laplace transform of this equation gives

$$\lambda\int dsp(s,\lambda)h_0(v,s)-\int dsp(s,0)h_0(v,s)- = -q(\lambda)\frac{\partial}{\partial s}p(s,t')h_0(v,s)-q(\lambda)p(0,\lambda)h_0(v,0). \quad \text{(B6)}$$

We have to require the initial condition

$$p(s,0)=\delta(s), \quad \text{(B7)}$$

which leads us to the relation

$$Q(\lambda)p(0,\lambda)=1. \quad \text{(B8)}$$

As a consequence, we obtain

$$\frac{\partial}{\partial t}\int dsp(s,t)=-\int ds\int_0^t dt' q(t-t')\frac{\partial}{\partial s}p(s,t') \quad \text{(B9)}$$

with the initial condition

$$\int_0^t dt' q(t-t')p(0,t')=\delta(t). \quad \text{(B10)}$$

This equation can easily be solved using Laplace transforms. We obtain

$$p(s,\lambda)=\frac{1}{q(\lambda)}e^{-\lambda/q(\lambda)s}, \quad \text{(B11)}$$

where we have taken into account the initial condition (B10).

For the fractional case

$$q(\lambda)=\lambda^{1-\alpha} \quad \text{(B12)}$$

this leads to

$$p(s,\lambda)=\lambda^{\alpha-1}e^{-\lambda^\alpha s}. \quad \text{(B13)}$$

Laplace inversion yields

$$p(s,t)=\frac{1}{\alpha}\frac{t}{s^{1+1/\alpha}}L_\alpha\left(\frac{t}{s^{1/\alpha}}\right). \quad \text{(B14)}$$

Thereby, $L_\alpha(x)$ is a one-sided Lévy stable probability distribution of order α which is defined in terms of its Laplace transform

$$L_\alpha(\lambda)=e^{-\lambda^\alpha}. \quad \text{(B15)}$$

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